1) The equilibrium point 0 of the system given by Eq. (5) is asymptotically stable at some time $t_0 \geq 0$.

2) The equilibrium point 0 of the system given by Eq. (6) is uniformly asymptotically stable over the interval $[0,\infty)$.

III. Examples

In this section we illustrate the efficacy of our theorem through two simple examples.

Example 1

Consider the simple example of the single-degree-of-freedom mass spring damper system

$$m\ddot{x} + c\dot{x} + kx = 0, \qquad m, c, k > 0 \tag{12}$$

where m, c, and k are the mass, damping, and stiffness of the system. We consider the analytic Lyapunov function V where

$$V = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 \tag{13}$$

represents the total energy of the system and whose derivative is given by the relation $\dot{V} = -c\dot{x}^2 \leq 0$. We compute the higher-order derivatives of V and check that when $\dot{V}=0$, $\ddot{V}=0$, and $V^{(3)} = -2c\ddot{x}^2 = -2c(k/m)^2x^2 < 0$, $\forall x \neq 0$, where $V^{(3)}$ is the third derivative of V with respect to time. Clearly, from the sufficient conditions of our theorem, we can conclude the asymptotic stability of the equilibrium point $x=0$ of the system described by Eq. (12).

Example 2

Consider the damped Mathieu equation that represents a nonautonomous system with a period of 2π. The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_2 - (2 + \sin t)x_1 \tag{14}$$

We choose the Lyapunov function V as

$$V(t, x_1, x_2) = x_1^2 + \frac{x_2^2}{2 + \sin t} \tag{15}$$

which is analytic in x_1, x_2, and t. Calculating $\dot{V}$, we find that

$$\dot{V} = -x_2^2 g(t), \qquad g(t) \triangleq \frac{4 + 2\sin t + \cos t}{(2 + \sin t)^2} \tag{16}$$

Thus $\dot{V} \leq 0$ for all t, (x_1, x_2) and $\dot{V}=0$ if and only if $x_2 = 0$. We compute the higher-order derivatives of V and find that when $\dot{V}=0$ or $x_2=0$, then $\ddot{V}=0$ and

$$V^{(3)} = -2\dot{x}_2^2 g(t) = -2(2 + \sin t)^2 x_1^2 g(t) < 0$$

$$\forall \boldsymbol{x} \triangleq (x_1 x_2)^T \neq 0 \tag{17}$$

It therefore follows from our theorem that the system described by Eq. (14) is uniformly asymptotically stable in the neighborhood of the equilibrium point.

IV. Conclusion

An asymptotic stability theorem for autonomous systems and periodic nonautonomous systems was developed that provides sufficient conditions to conclude asymptotic stability when the first derivative of the Lyapunov function vanishes. This theorem is more versatile than the well-known LaSalle's theorem because it does not require us to sort out the maximum invariant set. The theorem is applicable to analytic Lyapunov functions and is specially useful when higher-order derivatives of the Lyapunov function are easy to compute. The efficacy of our theorem was shown through two simple examples.

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Exact Closed-Form Solution of Generalized Proportional Navigation

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I. Introduction

PROPORTIONAL navigation has been widely used as the guidance scheme in the homing phase of flight for most missile systems. In pure proportional navigation (PPN), the commanded acceleration is applied in the direction normal to pursuer's velocity, and its magnitude is proportional to the angular rate of line of sight (LOS) between pursuer and its target.[1-4] In traditional true proportional navigation (TPN), the commanded acceleration is applied in a direction normal to the LOS, and its magnitude is proportional to the LOS rate.[5,6] Then a modified TPN is submitted, in which the commanded acceleration is applied in a direction normal to the LOS, and its magnitude is proportional to the product of LOS rate and closing speed between pursuer and target.[7,8] Furthermore, generalized proportional navigation (GPN) and ideal proportional navigation (IPN) were presented recently, in which the commanded acceleration is applied with a fixed bias angle to the direction normal to LOS and normal to the relative velocity between pursuer and target, respectively.[9-11]

In GPN, some solutions were previously obtained for a nonmaneuvering target that seem incomplete.[9,10] In this Note, we try to derive an exact and complete closed-form solution of GPN with a maneuvering target, which is much more general and comprehensive than those obtained before. Then a special case of target maneuver is discussed to easily illustrate the effect of a target maneuver. It can be solved as a function of deflection angle of LOS, in general. Some important and significant characteristics related to the system performance, such as capture capability and energy cost, are investigated and discussed in detail.

II. Closed-Form Solution

Consider a pursuer of speed V_M and a maneuvering target with speed V_T in exoatmospheric flight under the guidance law of GPN. The commanded acceleration is given with a bias angle β to the direction normal to LOS and its magnitude is proportional to LOS rate, i.e.,

$$\boldsymbol{a}_c = -\lambda v_0 \dot{\theta}(\cos\beta \boldsymbol{e}_\theta + \sin\beta \boldsymbol{e}_r) \tag{1}$$

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Here, λ is the effective proportional navigation constant and v_0 is the magnitude of initial relative velocity. The relative motion can be described in a polar coordinate system as

$$v = V_M - V_T = v_r e_r + v_\theta e_\theta \tag{2}$$

Then the equations of relative motion can be written as

$$v_r' = v_\theta - \lambda v_0 \sin\beta - f_r(\theta) \tag{3a}$$

$$v_\theta' = -v_r - \lambda v_0 \cos\beta - f_\theta(\theta) \tag{3b}$$

where primes denote differentiation with respect to θ; $f_r(\theta)$ and $f_\theta(\theta)$ correspond to target maneuver in two components of the polar coordinate, respectively, and are given as continuous functions of θ. Then, the general solutions of Eqs. (3) are

$$v_r = v_2\cos(\phi_2-\theta) - \lambda v_0\cos\beta + F_r(\theta_f-\theta) \tag{4a}$$

$$v_\theta = v_2\sin(\phi_2-\theta) + \lambda v_0\sin\beta + F_\theta(\theta_f-\theta) \tag{4b}$$

with

$$F_r(\theta_f-\theta) = \int_0^{\theta_f-\theta} \left[\cos(\theta_f-\theta-\alpha) f_r(\theta_f-\alpha) - \sin(\theta_f-\theta-\alpha) f_\theta(\theta_f-\alpha)\right] \mathrm{d}\alpha$$

$$F_\theta(\theta_f-\theta) = \int_0^{\theta_f-\theta} \left[\sin(\theta_f-\theta-\alpha) f_r(\theta_f-\alpha) + \cos(\theta_f-\theta-\alpha) f_\theta(\theta_f-\alpha)\right] \mathrm{d}\alpha$$

Here θ_f is the final deflection angle of LOS until intercept. Then the integration constant v_2 and ϕ_2 can be computed from initial condition, i.e.,

$$v_2 = \sqrt{\left[v_{r_0} + \lambda v_0\cos\beta - F_r(\theta_f)\right]^2 + \left[v_{\theta_0} - \lambda v_0 \sin\beta - F_\theta(\theta_f)\right]^2} \tag{5a}$$

$$\phi_2 = \tan^{-1}\frac{v_{\theta_0} - \lambda v_0\sin\beta - F_\theta(\theta_f)}{v_{r_0} + \lambda v_0\cos\beta - F_r(\theta_f)} \tag{5b}$$

where v_{r_0} and v_{θ_0} are two components of v_0 in the polar coordinate. Also, the thermal constraint $v_\theta(\theta_f) = 0$ must be satisfied for effective intercept, i.e.,

$$\theta_f = \phi_2 + \delta_2 \tag{6}$$

with $\delta_2 = \sin^{-1}[(\lambda v_0/v_2)\sin\beta]$. Then v_2, ϕ_2, and θ_f can be solved from Eqs. (5) and (6). The capture criterion can be obtained as the following inequality:

$$\lambda v_0 \cos\beta / v_2 \cos\delta_2 > 1 \tag{7a}$$

or

$$\lambda v_0 / v_2 > 1 \tag{7b}$$

The total cumulative velocity increment required can be expressed as

$$\Delta V = \int_0^T |a_c|\, \mathrm{d}t = \lambda v_0 |\theta_f| = \lambda v_0 |\phi_2 + \delta_2| \tag{8}$$

which is related to the corresponding propellant mass required for effective intercept in exoatmospheric flight. The relation between range-to-go r and LOS deflection angle θ can be derived as

$$r = r_0 \exp\left[\int_0^\theta \frac{v_r}{v_\theta}\,\mathrm{d}\alpha\right] = r_0 \exp\left[\int_0^\theta \frac{v_2\cos(\phi_2-\alpha) - \lambda v_0\cos\beta + F_r(\theta_f-\alpha)}{v_2\sin(\phi_2-\alpha) + \lambda v_0\sin\beta + F_\theta(\theta_f-\alpha)}\,\mathrm{d}\alpha\right] \tag{9}$$

and the response of LOS rate can be also obtained as

$$\dot\theta = \frac{v_\theta}{r} = \dot\theta_0 \frac{\left[v_2\sin(\phi_2-\theta) + \lambda v_0\sin\beta + F_\theta(\theta_f-\theta)\right] \Big/ \left[v_2\sin\phi_2 + \lambda v_0 \sin\beta + F_\theta(\theta_f)\right]}{\exp\left[\int_0^\theta \frac{v_2\cos(\phi_2-\alpha) - \lambda v_0\cos\beta + F_r(\theta_f-\alpha)}{v_2\sin(\phi_2-\alpha)+\lambda v_0\sin\beta + F_\theta(\theta_f-\alpha)}\,\mathrm{d}\alpha\right]} \tag{10}$$

Now a special case of target maneuver is considered to easily illustrate the system performance. Let $f_r = -cv_0\sin\beta$ and $f_\theta = -cv_0\cos\beta$, i.e., target maneuvers in the same direction as the commanded acceleration and $c(>0)$ is the factor of target maneuver. Thus, the solution in this case is simply

$$v_r = v_3\cos(\phi_3-\theta) - (\lambda-c)v_0\cos\beta \tag{11a}$$

$$v_\theta = v_3\sin(\phi_3-\theta) + (\lambda-c)v_0\sin\beta \tag{11b}$$

with

$$v_3 = v_0\sqrt{\left[\sin\phi_0 - (\lambda-c)\sin\beta\right]^2 + \left[\cos\phi_0 + (\lambda-c)\cos\beta\right]^2}$$

$$\phi_3 = \tan^{-1}\frac{\sin\phi_0 - (\lambda-c)\sin\beta}{\cos\phi_0 + (\lambda-c)\cos\beta}$$

and the capture criterion can be obtained as

$$\left[(\lambda-c)v_0/v_3\right] > 1 \tag{12a}$$

or

$$\lambda > \left[1/-2\cos(\phi_0+\beta)\right] + c \tag{12b}$$

where ϕ_0 is the angle between LOS and initial relative velocity. Therefore, a larger λ is required with a larger target maneuver for effective intercept. The cumulative velocity increment required is

$$\Delta V = \int_0^T |a_c|\,\mathrm{d}t = \lambda v_0|\theta_f| = \lambda v_0|\phi_3 + \delta_3| \tag{13}$$

with $\delta_3 = \sin^{-1}\{[(\lambda-c)v_0/v_3]\sin\beta\}$. Then a larger energy cost is induced by a larger target maneuver. The response of range-to-go and LOS rate can be written as

$$r = r_0\exp\left[\int_0^\theta \frac{v_r}{v_\theta}\,\mathrm{d}\alpha\right] = r_0\left[\frac{\sin ½(\phi_3+\delta_3-\theta)}{\sin ½(\phi_3+\delta_3)}\right]^{\frac{(\lambda-c)v_0}{v_3}\frac{\cos\beta}{\cos\delta_3}-1} \times \left[\frac{\cos ½(\phi_3-\delta_3)}{\cos ½(\phi_3-\delta_3-\theta)}\right]^{\frac{(\lambda-c)v_0}{v_3}\frac{\cos\beta}{\cos\delta_3}+1} \tag{14}$$

and

$$\dot\theta = \frac{v_\theta}{r} = \dot\theta_0\left[\frac{\sin ½(\phi_3+\delta_3-\theta)}{\sin ½(\phi_3+\delta_3)}\right]^{2-\frac{(\lambda-c)v_0}{v_3}\frac{\cos\beta}{\cos\delta_3}} \times \left[\frac{\cos ½(\phi_3-\delta_3-\theta)}{\cos ½(\phi_3-\delta_3)}\right]^{2+\frac{(\lambda-c)v_0}{v_3}\frac{\cos\beta}{\cos\delta_3}} \tag{15}$$

respectively. In this typical example, it shows that the LOS rate approaches zero when $[(\lambda - c)v_0/v_3](\cos\beta/\cos\delta_3) < 2$ and approaches infinity when $[(\lambda - c)v_0/v_3](\cos\beta/\cos\delta_3) > 2$ during intercept period.

III. Discussion

For the case of a maneuvering target, in general, a larger proportional navigation constant is required for effective intercept of the target and a high energy cost is induced during the intercept period. It is simply illustrated with a typical example as described in the previous section. Thus, target maneuver decreases the capture area and increases the energy cost required for effective intercept of target, as depicted in Figs. 1 and 2, respectively. We find that from Eq. (15) the value of $[(\lambda - c)v_0/v_3)](\cos\beta/\cos\delta_3)$, which must be >1 for effective intercept of target, determines whether the LOS rate approaches zero or infinity during the intercept period. When $[(\lambda - c)v_0/v_3](\cos\beta/\cos\delta_3) > 2$, which can be transformed to $-\cos(\phi_0+\beta) > \{[3(\lambda-c)^2\cos^2\beta + 4]/8(\lambda - c)\}$, the final LOS rate approaches infinity. When $[(\lambda - c)v_0/v_3](\cos\beta/\cos\delta_3) < 2$, it approaches zero until intercept. Thus, the capture area can be separated into two parts: the LOS rate approaches zero in one part and approaches infinity in the other, as depicted in Fig. 3. As a result of the limitation of commanded acceleration in an actual application, the proportional navigation constant must be selected appropriately to avoid the LOS rate approaching infinity for effective intercept of the target. For the case of a nonmaneuvering target, the solutions can be simply derived from the previous section with $c = 0$ and the capture criterion can be obtained as $\lambda > [1/-2\cos(\phi_0+\beta)]$, which is more comprehensive than those obtained before with a different definition of λ.[9,10] The capture area is affected by the bias angle β, where $|\phi_0+\beta| > \pi/2$ must be satisfied, as shown in Fig. 1. The bias angle also affects the energy cost required for effective intercept of target, a higher energy cost is required with a larger bias angle. Finally, the solution of TPN can be simply obtained from that of GPN with $\beta = 0$, which is different from that of modified TPN.[8,12]

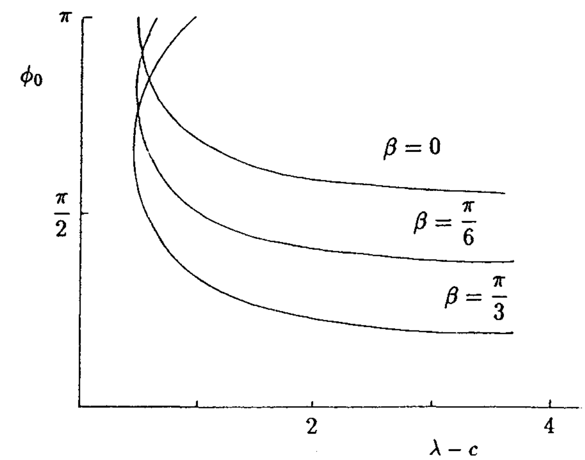

Fig. 1 Capture area of GPN with a maneuvering target.

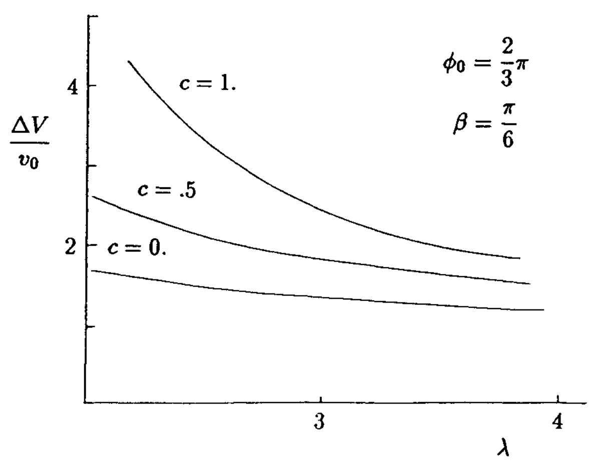

Fig. 2 Energy cost of GPN with a maneuvering target.

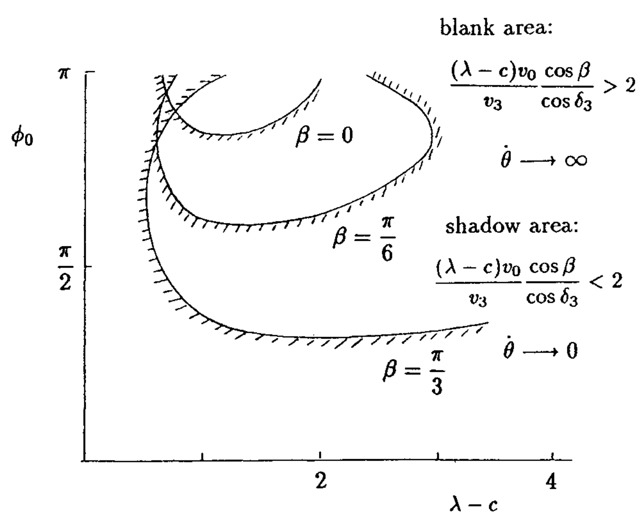

Fig. 3 Separation of capture area under GPN.

IV. Conclusion

In this Note, the exact and complete closed-form solutions of generalized proportional navigation with a maneuvering target are derived, which are more general and comprehensive than those obtained before. Also, some significant characteristics, such as capture capability and energy cost, are investigated and discussed in detail under the effect of bias of the commanded acceleration. Furthermore, a special case of target maneuver is introduced to easily describe the effect of target maneuver. The target maneuver decreases the capture area and increases the energy cost for effective intercept of the target.

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Optimal Pointing Control of Robotic Manipulators with State Inequality Constraints

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I. Introduction

THE problem of minimum-time pointing control of a planar, two-link robotic manipulator considers aligning the second link of the manipulator with a remote target point. This problem is quite different from the time-optimal control problems considered in Refs. 1–5 where the final position of the end effector is specified. Pointing a remote target imposes a geometric constraint on both of the joint angles of the manipulator. Therefore, the admissible control space is larger than those considered in Refs. 1–5. In addition, the Lagrange multipliers are also constrained by an equation at final time.

With the additional constraints on the final states and the Lagrange multipliers, optimal solutions have been generated in Ref. 6 for the pointing control problem of robotic manipulators. However, in space applications the second link cannot move freely without any structural interference. Because of this, we consider an inequality constraint to be imposed on the elbow joint angle. It is obvious that the minimum time will be increased. However, it is unclear how optimal solutions will be changed according to the state-variable inequality constraint. For example, both shoulder and elbow joint controls appear linearly in the performance index and the equation of motion; therefore, both controls may be bang-bang or singular depending on two switching functions and their derivatives. It has been shown in Ref. 8 that both controls cannot be singular simultaneously for the problem without state-variable inequality constraint. When the path stays on the state boundary, the two switching functions must be used together with one additional equality constraint to determine both controls. In this Note we will show that even with the state-variable inequality constraint there exists at least one nonsingular control for a rigid n-link robotic manipulator.

The computation of the optimal solutions can be simplified if the constrained and unconstrained arcs can be calculated separately. In Ref. 9 the separate computation has been shown possible for several conditions. We will show that the order of the state-variable inequality constraint for robotic manipulators is two and that the separate computation is impossible for this case. Furthermore, by using the constraint equations, we will show that, if the elbow control is nonsingular, the final point of the optimal path is isolated, if it touches the state boundary.

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II. Problem Formulation

A planar manipulator with two uniform rigid links for pointing control is shown in Fig. 1 where the shoulder joint is fixed to the base; L_1 is the length of the first link, L_2 the length of the second link, r the distance between the center of mass of the second link and the elbow joint, θ_1 the shoulder angle, θ_2 the elbow angle which is the relative angular rotation of the second link with respect to the first link, Ψ the target angle, and R the target distance scaled by L_1. The maximum allowable elbow joint angle is denoted by $\theta_{2\max}$. Equal bounds of magnitude are assumed for the torques T_1 and T_2

$$|T_i| \le T_{\max}, \qquad i = 1, 2 \tag{1}$$

and the elbow joint angle $\theta_2(t)$ is constrained by

$$|\theta_2(t)| \le \theta_{2\max} \tag{2}$$

Define the state variables

$$x_1(t) = \theta_1(t), \qquad x_2(t) = \dot\theta_1(t)$$
$$x_3(t) = \theta_2(t), \qquad x_4(t) = \dot\theta_2(t) \tag{3}$$

and the nondimensionalized control variables

$$u_i(t) = T_i(t)/T_{\max}, \qquad i = 1, 2 \tag{4}$$

with the control bounds set to $+1$ and -1

$$|u_i(t)| \le 1, \qquad i = 1, 2 \tag{5}$$

We write the dynamic equations as

$$\dot x_1 = x_2 \tag{6}$$

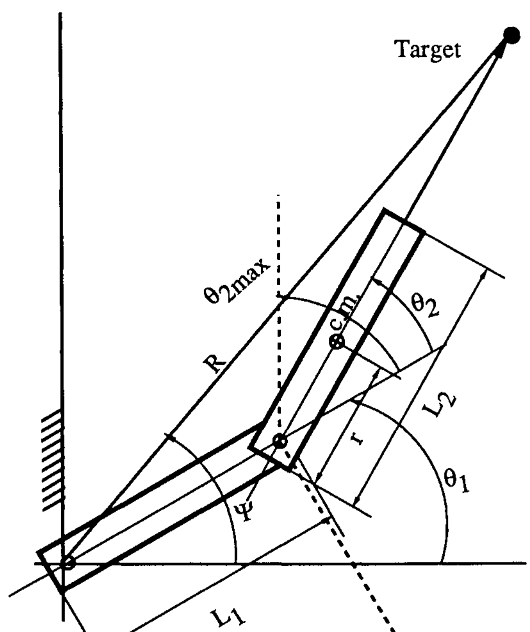

Fig. 1 Two-link robotic manipulator with elbow constraint.